# Potentials and Reduced Games for Share Functions

van den Brink, J.R.; van der Laan, G.

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René van den Brink[*]

Gerard van der Laan[‡]

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Correspondence to:
René van den Brink
Department of Econometrics
Tilburg University
P.O. Box 90153
5000 LE Tilburg
The Netherlands

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[*]Department of Econometrics and CentER, Tilburg University, P.O. Box 90153, Tilburg, The Netherlands. This author is financially supported by the Netherlands Organization for Scientific Research (NWO), ESR-grant 510-01-0504.

[‡]Department of Econometrics and Tinbergen Institute, Free University, De Boelelaan 1105, Amsterdam, The Netherlands.

**Abstract**

A situation in which a finite set of players can obtain certain payoffs by cooperation can be described by a *cooperative game with transferable utilities* –or simply a TU-game. A *value function* for TU-games is a function that assigns to every game a distribution of the payoffs. A value function is *efficient* if for every game it exactly distributes the worth that can be obtained by all players cooperating together.

An approach to efficiently allocating the worth of the 'grand coalition' is using *share functions* which assign to every game a vector which components sum up to one such that every component is the corresponding players' share in the total payoff that is to be distributed among the players. In this paper we give some characterizations of a class of share functions containing the *Shapley share function* and the *Banzhaf share function* using generalizations of potentials and of Hart and Mas-Colell's reduced game property.

# 1 Introduction

A situation in which a finite set of players can obtain certain payoffs by cooperation can be described by a *cooperative game with transferable utilities* –or simply a TU-game– being a pair $(N, v)$, where $N = \{1, \ldots, n\}$ is a finite set of players and $v: 2^N \to \mathbb{R}$ is a *characteristic function* on $N$ such that $v(\emptyset) = 0$. We denote the collection of all TU-games by $\mathcal{G}$.

A *value function* on $\mathcal{C} \subset \mathcal{G}$ is a function $f$ that assigns to every $(N, v) \in \mathcal{C}$ an $n$-dimensional real vector $f(N, v) \in \mathbb{R}^n$ representing a distribution of payoffs among the players. A value function $f$ is efficient on $\mathcal{C} \subset \mathcal{G}$ if for every game in $\mathcal{C}$ it exactly distributes the worth $v(N)$ of the 'grand coalition' over all players, i.e., if $\sum_{i \in N} f_i(N, v) = v(N)$ for every $(N, v) \in \mathcal{C}$. An example of an efficient value function is the *Shapley value* (Shapley (1953)), and an example of a value function that is not efficient is the *Banzhaf value* (Banzhaf (1965)) which is characterized in, e.g., Lehrer (1988) and Haller (1994). Since the Banzhaf value is not efficient it is not adequate in allocating the worth $v(N)$ of the 'grand coalition'. In order to allocate $v(N)$ according to the Banzhaf value van den Brink and van der Laan (1998a) characterized the *normalized Banzhaf value* which distributes the worth $v(N)$ proportional to the

Banzhaf values of the players. This normalized Banzhaf value does not satisfy some important properties that are used in characterizing the Banzhaf value.

An alternative approach to efficiently allocating the worth $v(N)$ of the 'grand coalition' is the concept of *share functions* as introduced in van der Laan and van den Brink (1998). A *share vector* for game $(N, v) \in \mathcal{G}$ is an $n$-dimensional real vector $\rho \in \mathbb{R}^n$ such that $\sum_{i \in N} \rho_i = 1$. Here $\rho_i$ is player $i$'s share in the total payoff that is to be distributed among the players. A *share function* on $\mathcal{C} \subset \mathcal{G}$ is a function that assigns to every $(N, v) \in \mathcal{C}$ exactly one share vector $\rho(N, v) \in \mathbb{R}^n$.

The share function corresponding to the Shapley value defined on the class of games $(N, v)$ for which $v(N) \neq 0$ is the *Shapley share function* which is obtained by dividing the Shapley value of each player by the sum of the Shapley values of all players (being equal to $v(N)$ since the Shapley value is efficient). Similarly, the *Banzhaf share function* is obtained by dividing the Banzhaf or normalized Banzhaf value by the corresponding sum of payoffs over all players. Note that, although the Banzhaf and normalized Banzhaf value are very different as argued above, they correspond to the same Banzhaf share function.

In van der Laan and van den Brink (1998) a class of share functions that generalizes the Shapley- and the Banzhaf share function is characterized. Purpose of this paper is to give two other characterizations of this class. First we generalize the concept of *potential function* as introduced in Hart and Mas-Colell (1988, 1989), yielding a class of potential functions such that their *normalized marginal functions* each correspond to a share function in the class of share functions mentioned above. Using this we also show how these share functions can be obtained from one another by adequately transforming games. In particular, we show how each share function in this class can be obtained as the Shapley share function of a transformed game.

Using this generalized concept of a potential function we give another characterization of the class of share functions considered. First we adapt the reduced games by Hart and Mas-Colell (1988, 1989) in characterizing the Shapley value, and Dragan (1996a,b) in characterizing the Banzhaf value to reduced games for share functions. Then we generalize these reduced games and use these to characterize other share functions.

The paper is organized as follows. In Section 2 we give some preliminaries on TU-games. In particular, we discuss the class of share functions and its characterization mentioned above. In Section 3 we characterize this class of share functions by considering generalized potential functions and their normalized marginal functions. In Section 4 we give a characterization of the class of share functions considered using reduced games. Finally, in Section 5 we make some concluding remarks on the relation between share functions and share mappings being mappings $R$ that assign to every $(N,v) \in \mathcal{C} \subset \mathcal{G}$ a set of share vectors $R(N,v) \subset \mathbb{R}^n$.

# 2 Preliminaries on TU-games and share functions

In this section we give some preliminary concepts and definitions on cooperative games. For given $N$ and nonempty $T \subset N$ the *unanimity game* $(N, u^T)$ is given by $u^T(E) = 1$ if $T \subset E$ and $u^T(E) = 0$ otherwise, $E \subset N$.[1] From Harsanyi (1959) we know that the characteristic function $v$ of a game $(N,v)$ can be expressed as a linear combination of the characteristic functions of the unanimity games $(T, u^T)$, $T \subset N$, by $v = \sum_{T \subset N} \Delta_v(T) u^T$ with $\Delta_v(T)$ the *dividend* of coalition $T \subset N$ given by $\Delta_v(T) = \sum_{E \subset T} (-1)^{(|T|-|E|)} v(E)$, where $|S|$ denotes the number of elements of the set $S$.

A TU-game $(N,v)$ is called *monotone* if $v(E) \leq v(F)$ for all $E \subset F \subset N$ and it is called *convex* if for every pair $E,\ F \subset N$ it holds that $v(E \cup F) + v(E \cap F) \geq v(E) + v(F)$. Observe that any unanimity game is monotone and convex. For a given game $(N,v) \in \mathcal{G}$ and given $T \subset N$, the *restriction* of $(N,v)$ to $T$ is denoted by the *subgame* $(T, v_T)$ and is given by $v_T(E) = v(E)$ for all $E \subset T$. The class $\mathcal{C} \subset \mathcal{G}$ is called *subgame closed* if for every $(N,v) \in \mathcal{C}$ and every $T \subset N$ it holds that $(T, v_T) \in \mathcal{C}$. Examples of subgame closed classes of games are the class of all games $\mathcal{G}$, the class of all monotone games, and the class of all convex games. Note that a class of games with a fixed player set is not subgame closed.

A game $(N,v)$ is called a *null game* if $v = v^0$ with $v^0(E) = 0$ for all $E \subset N$. To conclude these preliminaries, let $\mu\colon \mathcal{G} \to \mathbb{R}$ be a function assigning a real value to any

[1] Note that we ignore the unanimity games $(N, u^{\emptyset})$. In the paper, when we speak about unanimity games we mean unanimity games $(N, u^T)$ with $T \neq \emptyset$.

game $(N, v) \in \mathcal{G}$. The function $\mu: \mathcal{G} \to \mathbb{R}$ is *positive* on $\mathcal{C} \subset \mathcal{G}$ if $\mu(N, v) > 0$ for all $(N, v) \in \mathcal{C}$, and it is called *zero* on $\mathcal{C} \subset \mathcal{G}$ if $\mu(N, v) = 0$ for all $(N, v) \in \mathcal{C}$. By $\mathcal{G}_\mu^+ \subset \mathcal{G}$, respectively $\mathcal{G}_\mu^0 \subset \mathcal{G}$, we denote the class of games on which $\mu$ is positive, respectively zero. Moreover, we define $\mathcal{G}_\mu = \mathcal{G}_\mu^+ \cup \mathcal{G}_\mu^0$, i.e. $\mu(N, v) \geq 0$ for all $(N, v) \in \mathcal{G}_\mu \subset \mathcal{G}$. We call a function $\mu: \mathcal{G} \to \mathbb{R}$ *additive* on $\mathcal{C}$ if for every pair of games $(N, v), (N, w) \in \mathcal{C}$ such that[2] $(N, v + w) \in \mathcal{C}$ it holds that $\mu(N, v + w) = \mu(N, v) + \mu(N, w)$. A function $\mu: \mathcal{G} \to \mathbb{R}$ is *linear* on $\mathcal{C}$ if it is additive on $\mathcal{C}$ and for every $(N, v) \in \mathcal{C}$ and $c \in \mathbb{R}$ such that $(N, cv) \in \mathcal{C}$ it holds that $\mu(N, cv) = c\mu(N, v)$. Finally, we call $\mu: \mathcal{G} \to \mathbb{R}$ *symmetric* on $\mathcal{C}$ if for every $(N, v) \in \mathcal{C}$, every pair of symmetric players $i, j$ in $(N, v)$ [3] and every $E \subset N$, $E \supset \{i, j\}$, such that the subgames $(E \setminus \{i\}, v_{E\setminus\{i\}})$ and $(E \setminus \{j\}, v_{E\setminus\{j\}})$ are in $\mathcal{C}$, it holds that $\mu(E \setminus \{i\}, v_{E\setminus\{i\}}) = \mu(E \setminus \{j\}, v_{E\setminus\{j\}})$.

We now recall some well-known solution concepts for cooperative games that are mentioned in the introduction. The Shapley value (Shapley (1953)) is the value function $Sh$ given by

$$Sh_i(N, v) = \sum_{\substack{E \subset N \\ E \ni i}} \frac{(|E| - 1)!(n - |E|)!}{n!} m_E^i(N, v) \text{ for all } i \in N,$$

where $m_E^i(N, v) = v(E) - v(E \setminus \{i\})$ is the *marginal contribution* of player $i$ to coalition $E \subset N$ in $(N, v) \in \mathcal{G}$. As mentioned in the introduction the Shapley value is an efficient value function.

A value function that is not efficient is the Banzhaf value (Banzhaf (1965)) being the value function $\beta$ given by

$$\beta_i(N, v) = \frac{1}{2^{n-1}} \sum_{\substack{E \subset N \\ E \ni i}} m_E^i(N, v) \text{ for all } i \in N.$$

In order to efficiently allocate $v(N)$ according to the Banzhaf value the *normalized*

[2] For a pair of games $(N, v), (N, w) \in \mathcal{G}$ the game $(N, v + w)$ is given by $(v + w)(E) = v(E) + w(E)$ for all $E \subset N$.

[3] Players $i, j \in N$ are *symmetric* in $(N, v) \in \mathcal{G}$ if $v(E \setminus \{i\}) = v(E \setminus \{j\})$ for all $E \subset N$ with $E \supset \{i, j\}$.

*Banzhaf value* $\widehat{\beta}$ given by

$$\widehat{\beta}(N,v) = \frac{v(N)}{\sum_{j\in N}\beta_j(N,v)}\beta(N,v)$$

can be used. Thus, the normalized Banzhaf value allocates $v(N)$ proportional to the Banzhaf values of the players.

A general approach to efficiently allocating payoffs in TU-games is using share functions which are introduced in van der Laan and van den Brink (1998). A *share function* on a set of games $\mathcal{C} \subset \mathcal{G}$ is a function $\rho$ that assigns to every game $(N,v) \in \mathcal{C}$ an $n$-dimensional real vector $\rho(N,v) \in \mathbb{R}^n$ such that the shares assigned to the players sum up to one for every game in $\mathcal{C}$, i.e. $\sum_{i\in N}\rho_i(N,v) = 1$ for all $(N,v) \in \mathcal{C}$. The $i^{th}$ component is the share of player $i \in N$ in the value to be distributed, e.g., in $v(N)$. Three properties that can be satisfied by such share functions are the following[4].

The first two properties are similar to the null player and symmetry properties for value functions. The share function $\rho$ satisfies the *null player property* on $\mathcal{C}$ if for every $(N,v) \in \mathcal{C}$ and every *null player* [5] $i$ in $(N,v)$ it holds that $\rho_i(N,v) = 0$. Share fuction $\rho$ satisfies *symmetry* on $\mathcal{C}$ if for every $(N,v) \in \mathcal{C}$ and every pair $i,j$ of *symmetric* players in $(N,v)$ it holds that $\rho_i(N,v) = \rho_j(N,v)$. Finally, for some function $\mu\colon \mathcal{G} \to \mathbb{R}$, the share function $\rho$ satisfies $\mu$-additivity on $\mathcal{C}$ if for every pair of games $(N,v),(N,w) \in \mathcal{C}$ such that $(N,v+w) \in \mathcal{C}$ it holds that $\mu(N,v+w)\rho(N,v+w) = \mu(N,v)\rho(N,v) + \mu(N,w)\rho(N,w)$. This last property is a generalization of the additivity property which is obtained by taking $\mu(N,v) = 1$ for all $(N,v) \in \mathcal{G}$. Although additivity is a reasonable property of value functions it does not make sense for share functions. However, a share function that satisfies $\mu$-additivity for an additive $\mu$-function satisfies some kind of weighted additivity property in the sense that the shares assigned to the sum game of two games is a convex combination of the shares assigned to the two separate games. This can easily be seen by rewriting $\mu$-addivity for an additive $\mu$-function as $\rho(N,v+w) = \frac{\mu(N,v)}{\mu(N,v)+\mu(N,w)}\rho(N,v) + \frac{\mu(N,w)}{\mu(N,v)+\mu(N,w)}\rho(N,w)$. So, $\mu$ determines the weights of the

[4]In van der Laan and van den Brink (1998) *efficient shares* (meaning that the components of $\rho_i(N,v)$ sum up to one for all $(N,v) \in \mathcal{C}$ is taken as a fourth axiom. In this paper we have taken this into our definition of a share function.

[5]Player $i \in N$ is a *null player* in $(N,v) \in \mathcal{G}$ if $v(E) = v(E \setminus \{i\})$ for all $E \subset N$.

games in this convex combination. What weights are appropriate depends on the application we have in mind. Here, we only require $\mu$ to be additive.

The following theorem[6] characterizes a class of share functions on subclasses of games $\mathcal{C} \subset \mathcal{G}$ containing all positively scaled unanimity games $(N, \alpha u^T)$, $T \subset N$, $\alpha > 0$, i.e. $\alpha u^T(E) = \alpha$ if $T \subset E$, and $\alpha u^T(E) = 0$ otherwise. Examples of classes of games that contain all positively scaled unanimity games are the class of all games $\mathcal{G}$, the class of all monotone games, and the class of all convex games.

**Theorem 2.1 (van der Laan and van den Brink (1998))**
(i) *Let $\mu: \mathcal{G} \to \mathbb{R}$ be positive and symmetric on a subclass $\mathcal{C} \subset \mathcal{G}$ that contains all positively scaled unanimity games. Then there exists a unique share function $\rho^\mu$ on $\mathcal{C}$ satisfying the null player property, symmetry and $\mu$-additivity if and only if $\mu$ is additive on $\mathcal{C}$.*
(ii) *For given positive vectors $\omega^n \in \mathbb{R}^n_+$, $n \in \mathbb{N}$, for all $n \in \mathbb{N}$, let the function $\mu: \mathcal{G} \to \mathbb{R}$ be defined by $\mu(N, v) = \sigma^{\omega^n}(N, v)$, where $\sigma^{\omega^n}: \mathcal{G} \to \mathbb{R}$ is given by*

$$\sigma^{\omega^n}(N, v) = \sum_{i \in N} \sum_{E \ni i} \omega^n_{|E|} m^i_E(N, v).$$

*Then the share function $\rho^{\omega^n}$ on $\mathcal{G}^+_{\sigma^{\omega^n}}$ given by*

$$\rho^{\omega^n}_i(N, v) = \frac{\sum_{\substack{E \subset N \\ E \ni i}} \omega^n_{|E|} m^i_E(N, v)}{\sigma^{\omega^n}(N, v)} \text{ for every } i \in N,$$

*is the unique share function satisfying the null player property, symmetry, and $\sigma^{\omega^n}$-additivity on $\mathcal{G}^+_{\sigma^{\omega^n}}$.*

The second part of the theorem shows that any choice of positive weights on the marginal contributions (with equal weights assigned to coalitions of equal size) defines a share function satisfying the null player property, symmetry and $\sigma^{\omega^n}$-additivity on $\mathcal{G}^+_{\sigma^{\omega^n}}$. Note that all functions $\sigma^{\omega^n}$ are positive on all positively scaled unanimity games.

[6] In van der Laan and van den Brink (1998) results are stated more general for classes of games with fixed player set.

Alternatively, for $\beta^n = (\beta_1^n, \ldots, \beta_n^n) \in \mathbb{R}^n$, $n \in \mathbb{N}$, a vector of real numbers, consider the function $\mu: \mathcal{G} \to \mathbb{R}$ defined by $\mu(N,v) = \eta^{\beta^n}(N,v)$, where $\eta^{\beta^n}: \mathcal{G} \to \mathbb{R}$ is given by

$$\eta^{\beta^n}(N,v) = \sum_{E \subset N} \beta_{|E|}^n v(E), \tag{1}$$

i.e. $\beta^n$ is a vector of weights putted to the worths $v(E)$, $E \subset N$, of the characteristic function. In van der Laan and van den Brink (1998) it is shown that $\sigma^{\omega^n}(N,v)$ is equal to $\eta^{\beta^n}(N,v)$ for all $(N,v) \in \mathcal{G}$ if and only if

$$\begin{aligned} \beta_n^n &= n\omega_n^n \\ \beta_t^n &= t\omega_t^n - (n-t)\omega_{t+1}^n, \ t = n-1, \ldots, 1. \end{aligned} \tag{2}$$

Observe that for arbitrarily chosen positive numbers $\omega_t^n$, $t = 1, \ldots, n$, some of the corresponding values of $\beta_t^n$, $t = 1, \ldots, n$, may be negative. However, the functions $\sigma^{\omega^n}$ (respectively $\eta^{\beta^n}$) satisfy the properties of additivity and symmetry for any positive vector $\omega^n$ (respectively the corresponding vector $\beta^n$) of weights.

Examples of $\mu$ functions defined by a vector $\beta^n$ of weights are the function $\mu^S: \mathcal{G} \to \mathbb{R}$ given by $\mu^S(N,v) = v(N)$ (with $\beta_n^n = 1$ and $\beta_t^n = 0$ for $t = 1, \ldots, n-1$), and $\mu^B: \mathcal{G} \to \mathbb{R}$ given by $\mu^B(N,v) = \frac{1}{2^{n-1}} \sum_{E \subset N} (2|E| - n)v(E)$ (with $\beta_t^n = (2t - n)2^{-(n-1)}$, $t = 1, \ldots, n$). In van der Laan and van den Brink (1998) it is shown that the unique share function satisfying the properties stated in Theorem 2.1 with $\mu = \mu^S$ is the Shapley share function $\rho^S$ given by

$$\rho_i^S(N,v) = \frac{Sh_i(N,v)}{v(N)} \text{ for all } i \in N,$$

on the class of games $(N,v) \in \mathcal{G}$ with $v(N) \neq 0$, and the unique share function satisfying these properties with $\mu = \mu^B$ is the Banzhaf share function $\rho^B$ given by

$$\rho_i^B(N,v) = \frac{\beta_i(N,v)}{\sum_{j \in N} \beta_j(N,v)} = \frac{\overline{\beta}_i(N,v)}{\sum_{j \in N} \overline{\beta}_j(N,v)} = \frac{2^{n-1}\beta_i(N,v)}{\sum_{E \subset N} \sum_{i \in E} m_E^i(N,v)} \text{ for all } i \in N$$

on the class of games $(N,v)$ for which $\sum_{j \in N} \beta_j(N,v) \neq 0$. This can be seen by noting that taking $\beta_n^n = 1$ and $\beta_t^n = 0$ for $t = 1, \ldots, n-1$ and solving system (2) for the vector $\omega^n$ gives the Shapley weights $\omega_t^n = \frac{(t-1)!(n-t)!}{n!}$, $t = 1, \ldots, n$, which gives us the Shapley share function by applying Theorem 2.1.(ii). Also, taking $\beta_t^n = (2t - n)2^{-(n-1)}$, $t =$

$1, \ldots, n$ and solving system (2) for the vector $\omega^n$ gives the Banzhaf weights $\omega_t^n = 2^{-(n-1)}$ for $t = 1, \ldots, n$, which gives us the Banzhaf share function by applying part (ii) of Theorem 2.1.

Besides the Shapley and Banzhaf share functions, other share functions can be obtained by particular choices of weight vectors $\omega$. We illustrate this by two examples. The *Deegan-Packel share function* $\rho^{DP}$ is given by $\rho_i^{DP}(N, v) = \frac{\varphi_i^{DP}(N,v)}{\sum_{j \in N} \varphi_j^{DP}(N,v)}$, $i \in N$, where $\varphi^{DP}$ is the non-efficient *Deegan-Packel value* (Deegan and Packel (1979)) given by

$$\varphi_i^{DP}(N, v) = \sum_{\substack{E \subset N \\ E \ni i}} \frac{v(E)}{|E|} \text{ for all } i \in N.$$

This share function satisfies the axioms of symmetry and $\mu^T$-additivity with $\mu^T(N, v) = \sum_{E \subset N} v(E)$, i.e. $\mu^T$ measures the sum of the worths of all coalitions of $N$. But it does not satisfy the null player property and thus does not belong to the class of share functions discussed above. However, according to Theorem 2.1 there exists a unique share function on $\mathcal{G}^+_{\mu^T}$ satisfying symmetry, $\mu^T$-additivity and the null player property. Again, taking $\beta_t = 1$, $t = 1, \ldots, n$ and and solving system (3) for the vector $\omega$ gives the weights $\omega_n = \frac{1}{n}$ and $\omega_t = \frac{1+(n-t)\omega_{t+1}}{t}$, $t = n-1, \ldots, 1$. Then the share function $\rho^T$ is found by using these weights in part (ii) of Theorem 2.1.

Another alternative would be to take $\mu^M(N, v) = nv(N) - \sum_{i \in N} v(N \setminus \{i\}) = \eta^\beta(N, v)$ with weights $\beta_n = n$, $\beta_{n-1} = -1$ and $\beta_t = 0$, $t = 1, \ldots, n-2$, or equivalently $\mu^M(N, v) = \sigma^\omega(N, v)$ with weights $\omega_n = 1$ and $\omega_t = 0$, $t = n-1, \ldots, 1$. The corresponding share function $\rho^M$ on $\mathcal{G}^+_{\mu^M}$ satisfying the null player property, symmetry and $\mu^M$-additivity is obtained by applying the vector $\omega$ of weights in the second part of Theorem 2.1. (Although not all weights are positive, note that the class $\mathcal{G}^+_{\mu^M}$ contains all unanimity games, which is sufficient to apply Theorem 2.1.) The share function $\rho^M$ distributes the shares proportional to the marginal contributions $m_N^i(N, v)$ of the players to the 'grand coalition' $N$, and thus is related to the class of compromise values such as the $\tau$-*value* (Tijs (1981)) (Note that the $\tau$-share function $\rho^\tau$ which assigns shares proportional to the $\tau$-value satisfies the axioms of null player property and symmetry, but there is no additive $\mu$-function such that $\rho^\tau$ is $\mu$-additivite.)

We conclude this section by extending Theorem 2.1.(i) from subclasses of the class $\mathcal{G}_\mu^+$ of $\mu$-positive games to subclasses of the set $\mathcal{G}_\mu$, so allowing for games to which the function $\mu$ assigns the value zero. The next corollary follows immediately from part (i) of Theorem 2.1 by requiring that $\rho^\mu$ satisfies the *equal share property* in case $(N,v)$ is a game with $\mu(N,v)=0$, i.e. $\rho_i^\mu(N,v)=\frac{1}{n}$ for all $i\in N$ when $(N,v)\in\mathcal{C}\cap\mathcal{G}_\mu^0$.

**Corollary 2.2** *Let $\mu:\mathcal{G}\to\mathbb{R}$ be additive and symmetric on $\mathcal{G}_\mu$, and let $\mathcal{C}\subset\mathcal{G}_\mu$ be a subgame closed set containing all positively scaled unanimity games. Then there exists a unique share function $\rho^\mu$ on $\mathcal{C}$ satisfying (i) symmetry and $\mu$-additivity on $\mathcal{C}$, (ii) the null player property on $\mathcal{C}\cap\mathcal{G}_\mu^+$, and (iii) the equal share property on $\mathcal{C}\cap\mathcal{G}_\mu^0$*

In a similar way the second part of Theorem 2.1 can be extended to subclasses of $\mathcal{G}_\mu$.

# 3 Potential functions

In the previous section we discussed share functions, restated the characterization of a class of share functions from van der Laan and van den Brink (1998), and gave a definition of these share functions using marginal share vectors. In this section we take another approach to share functions by generalizing the concept of *potential functions* as introduced in Hart and Mas-Colell (1988, 1989). Hart and Mas-Colell define a *potential function* on a subclass $\mathcal{C}$ of $\mathcal{G}$ to be a function $P:\mathcal{C}\to\mathbb{R}$ satisfying $P(N,v)=0$ if $N=\emptyset$, and whenever $N\neq\emptyset$ it must hold that

$$\sum_{i\in N}\Big(P(N,v)-P(N\setminus\{i\},v_{N\setminus\{i\}})\Big)=v(N).$$

Hart and Mas-Colell show that there exists a unique potential function on $\mathcal{C}\subset\mathcal{G}$, when $\mathcal{C}$ is a subgame closed set of games. They also show that the vector function $DP$ on $\mathcal{C}$ of marginals defined by $DP_i(N,v)=P(N,v)-P(N\setminus\{i\},v_{N\setminus\{i\}})$, $i\in N$, is the Shapley value function. Using $\mu$-functions as introduced in the previous section we generalize the concept of potential function as follows.

**Definition 3.1** *Let $\mathcal{C} \subset \mathcal{G}$ be subgame closed. Then, for given $\mu: \mathcal{G} \to \mathbb{R}$, a function $P^\mu: \mathcal{C} \to \mathbb{R}$ is a $\mu$-***potential function** *on $\mathcal{C}$ if $P^\mu(N,v) = 0$ whenever $N = \emptyset$, and for every $(N,v) \in \mathcal{C}$ with $N \neq \emptyset$ it holds that*

$$\sum_{i \in N} \left( P^\mu(N,v) - P^\mu(N \setminus \{i\}, v_{N \setminus \{i\}}) \right) = \mu(N,v). \tag{3}$$

Clearly, $P^\mu$ is the Hart and Mas-Colell potential function when the function $\mu$ is taken to be the Shapley $\mu$-function $\mu^S$ which assigns to every TU-game $(N,v)$ the worth $v(N)$ of the grand coalition[7]. For given function $\mu: \mathcal{G} \to \mathbb{R}$, in the remaining of this section we restrict our analysis to subgame closed subsets $\mathcal{C}$ of the class of games $\mathcal{G}_\mu$. For $\mathcal{C} \subset \mathcal{G}_\mu$, we define $\mathcal{C}^+ = \mathcal{C} \cap \mathcal{G}_\mu^+$ and $\mathcal{C}^0 = \mathcal{C} \cap \mathcal{G}_\mu^0$. Now, for given $\mu$-potential function $P^\mu$ on a subgame closed set $\mathcal{C} \subset \mathcal{G}_\mu$ we define the *marginal function* $DP^\mu$ on $\mathcal{C}$ by

$$DP_i^\mu(N,v) = P^\mu(N,v) - P^\mu(N \setminus \{i\}, v_{N \setminus \{i\}}),\ i \in N, \tag{4}$$

and the *normalized marginal function* $NDP^\mu$ on $\mathcal{C}$ by

$$NDP_i^\mu(N,v) = \left\{ \begin{array}{cl} \frac{DP_i^\mu(N,v)}{\mu(N,v)} & \text{if } (N,v) \in \mathcal{C}^+ \\ \frac{1}{n} & \text{if } (N,v) \in \mathcal{C}^0 \end{array} \right\},\ i \in N. \tag{5}$$

We will prove that for given function $\mu: \mathcal{G} \to \mathbb{R}$ and subgame closed set $\mathcal{C} \subset \mathcal{G}_\mu$, the normalized marginal function on $\mathcal{C}$ characterizes the corresponding share function $\rho^\mu$ on $\mathcal{C}$ satisfying the conditions of Corollary 2.2 when the $\mu$-function satisfies an additional property, namely the property of null player independency. This property states that deleting a null player from a game does not change the value assigned by the $\mu$-function.

**Definition 3.2** *The function $\mu: \mathcal{G} \to \mathbb{R}$ is* **null player independent** *on $\mathcal{C} \subset \mathcal{G}$ if for every $(N,v) \in \mathcal{C}$ and every null player $i$ in $(N,v)$ such that $(N \setminus \{i\}, v_{N \setminus \{i\}}) \in \mathcal{C}$ it holds that $\mu(N,v) = \mu(N \setminus \{i\}, v_{N \setminus \{i\}})$.*

[7] Calvo and Santos (1997) consider value functions $\psi$ on the class of all games for which there exists a function $P_\psi$ from the class of games to the real numbers such that $\psi_i(N,v) = P_\psi(N,v) - P_\psi(N \setminus \{i\}, v_{N \setminus \{i\}})$. They show that such a $P_\psi$ exists if and only if $\psi(N,v) = \psi(N, v_\psi)$ where $v_\psi$ is the characteristic function which is determined by $\psi$ itself by assigning to every $E \subset N$ the sum $\sum_{i \in E} \psi_i(N,v)$.

Further, we only consider functions $\mu$ that assign the value zero to null games. We now state the following result.

**Theorem 3.3** *Let $\mu\colon \mathcal{G} \to \mathbb{R}$ with $\mu(N, v^0) = 0$ for every set of players $N$, be additive, symmetric, and null player independent on $\mathcal{G}_\mu$, and let $\mathcal{C} \subset \mathcal{G}_\mu$ be a subgame closed set containing all positively scaled unanimity games. Then there exists a unique $\mu$-potential function $P^\mu$ on $\mathcal{C}$. The corresponding normalized marginal function $NDP^\mu$ is equal to the unique share function $\rho^\mu$ on $\mathcal{C}$ satisfying the properties of Corollary 2.2.*

PROOF

Since $P^\mu(N, v) = 0$ if $N = \emptyset$, the potential $P^\mu(N, v)$, $n \geq 1$, is uniquely determined by recursively using equation (3) and is given by

$$P^\mu(N, v) = \frac{1}{n}\left(\mu(N, v) + \sum_{i \in N} P^\mu(N \setminus \{i\}, v_{N\setminus\{i\}})\right). \tag{6}$$

From the definition of the corresponding normalized marginal function given in equation (5) it follows that $NDP^\mu(N, v) = \rho^\mu(N, v) = \frac{1}{n}$ if $(N, v) \in \mathcal{C}^0$. To prove that $NDP^\mu(N, v) = \rho^\mu(N, v)$ for all $(N, v) \in \mathcal{C}$ it is sufficient to show that $NDP^\mu$ is a share function and satisfies the null player property on $\mathcal{C}^+$, symmetry and $\mu$-additivity, since Corollary 2.2 says that there is a unique share function satisfying these properties.

From the equations (3), (4) and (5) it follows immediately that the components of $NDP^\mu$ add up to one and thus it is a share function. To prove the null player property on $\mathcal{C}^+$, let $i \in N$ be a null player in $(N, v) \in \mathcal{C}$. First, take $N = \{i\}$ and hence $v = v^0$ because $i$ is a null player. From equation (3) and $P^\mu(\emptyset, v) = 0$ for all $v$ we obtain that

$$DP_i^\mu(N, v) = P^\mu(\{i\}, v) - P^\mu(\emptyset, v_\emptyset) = \mu(\{i\}, v) = \mu(\{i\}, v^0) = 0.$$

Hence, $DP_i^\mu(N, v) = 0$ when $n = 1$ and $i$ is a null player. Proceeding by induction assume that for some given integer $k \geq 1$ and for any game $(N', v) \in \mathcal{C}$ with $i \in N'$ a null player and $|N'| = k$, it holds that $DP_i^\mu(N', v) = 0$, and let $N$ be such that $N' \subset N$ and $n = k+1$. Using the induction hypotheses with $N' = N \setminus \{j\}$ for all $j \in N \setminus \{i\}$ we obtain that for given $n$-player game $(N, v)$ with the induced restricted $(n-1)$-player games $(N \setminus \{j\}, v_{N\setminus\{j\}})$ (in which player $i$ is a null player) it holds that

$$\begin{aligned} nDP_i^\mu(N,v) &= DP_i^\mu(N,v) + \sum_{j\in N\setminus\{i\}} DP_i^\mu(N,v) \\ &= DP_i^\mu(N,v) + \sum_{j\in N\setminus\{i\}} \Big(DP_i^\mu(N,v) - DP_i^\mu(N\setminus\{j\}, v_{N\setminus\{j\}})\Big) \\ &= P^\mu(N,v) - P^\mu(N\setminus\{i\}, v_{N\setminus\{i\}}) \\ &\quad + \sum_{j\in N\setminus\{i\}} \Big(P^\mu(N,v) - P^\mu(N\setminus\{j\}, v_{N\setminus\{j\}})\Big) \\ &\quad - \sum_{j\in N\setminus\{i\}} \Big(P^\mu(N\setminus\{i\}, v_{N\setminus\{i\}}) - P^\mu(N\setminus\{i,j\}, v_{N\setminus\{i,j\}})\Big) \\ &= \mu(N,v) - \mu(N\setminus\{i\}, v_{N\setminus\{i\}}). \end{aligned}$$

Hence, null player independency of $\mu$ implies that for every $(N,v) \in \mathcal{C}$ it holds that

$$DP_i^\mu(N,v) = \frac{1}{n}\Big(\mu(N,v) - \mu(N\setminus\{i\}, v_{N\setminus\{i\}})\Big) = 0,$$

when $i$ is a null player in $(N,v)$, and thus $NDP_i^\mu(N,v) = 0$ for $(N,v) \in \mathcal{C}^+$, which shows that the null player property is true on $\mathcal{C}^+$.

To prove the symmetry property we show that for every $(N,v) \in \mathcal{C}$ it holds that $NDP_i^\mu(N,v) = NDP_j^\mu(N,v)$, when $i$ and $j$ are two symmetric players in $(N,v)$. So, let $i,j \in N$ be two symmetric players in $(N,v) \in \mathcal{G}$. For $N = \{i,j\}$ it follows with $P^\mu(\emptyset, v) = 0$ and the symmetry of $\mu$ that $P^\mu(N\setminus\{i\}, v_{N\setminus\{i\}}) = P^\mu(\{j\}, v_{\{j\}}) = \mu(\{j\}, v_{\{j\}}) = \mu(\{i\}, v_{\{i\}}) = P^\mu(\{i\}, v_{\{i\}}) = P^\mu(N\setminus\{j\}, v_{N\setminus\{j\}})$. Hence, $NDP_i^\mu(N,v) = NDP_j^\mu(N,v)$. Proceeding by induction assume that for some given integer $k \geq 2$ and for any game $(N',v) \in \mathcal{C}$ with $i,j \in N'$ and $|N'| = k$, it holds that $P^\mu(N'\setminus\{i\}, v_{N'\setminus\{i\}}) = P^\mu(N'\setminus\{j\}, v_{N'\setminus\{j\}})$ and let $N$ be such that $N' \subset N$ and $n = k+1$. Using symmetry of $\mu$ and the induction hypotheses with $N' = N\setminus\{h\}$ for all $h \in N\setminus\{i,j\}$ we obtain that for given $n$-player game $(N,v)$ with the induced restricted $(n-1)$-player games $(N\setminus\{h\}, v_{N\setminus\{h\}})$ it holds that

$$\begin{aligned} P^\mu(N\setminus\{i\}, v_{N\setminus\{i\}}) &= \frac{1}{n-1}\left(\mu(N\setminus\{i\}, v_{N\setminus\{i\}}) + \sum_{h\in N\setminus\{i\}} P^\mu(N\setminus\{i,h\}, v_{N\setminus\{i,h\}})\right) \\ &= \frac{1}{n-1}\left(\mu(N\setminus\{j\}, v_{N\setminus\{j\}}) + \sum_{h\in N\setminus\{i\}} P^\mu(N\setminus\{j,h\}, v_{N\setminus\{j,h\}})\right) \end{aligned}$$

$$= \quad P^\mu(N \setminus \{j\}, v_{N\setminus\{j\}}).$$

So, for every $(N,v) \in \mathcal{C}$ and two symmetric players $i,j \in N$ it holds that $DP_i^\mu(N,v) = P^\mu(N,v) - P^\mu(N \setminus \{i\}, v_{N\setminus\{i\}}) = P^\mu(N,v) - P^\mu(N \setminus \{j\}, v_{N\setminus\{j\}}) = DP_j^\mu(N,v)$. Hence, $NDP_i^\mu(N,v) = NDP_j^\mu(N,v)$.

Finally we prove $\mu$-additivity of $NDP^\mu$. For $(N,v),\ (N,w) \in \mathcal{C}$ with $n=1$ it holds that $P^\mu(N,v+w) = \mu(N,v+w) = \mu(N,v)+\mu(N,w) = P^\mu(N,v)+P^\mu(N,w)$ by additivity of $\mu$. Proceeding by induction assume that for some given integer $k \geq 1$ and for any pair $(N',v),\ (N',w) \in \mathcal{C}$ with $|N'| = k$, it holds $P^\mu(N',v+w) = P^\mu(N',v) + P^\mu(N',w)$. Then for $N$ with $n = k+1$, it follows from the additivity of $\mu$ and the induction hypothesis for all $N' = N \setminus \{j\}$ for all $j \in N$, that

$$\begin{aligned}
P^\mu(N,v+w) &= \frac{1}{n}\left(\mu(N,v+w) + \sum_{i\in N} P^\mu(N\setminus\{i\}, (v+w)_{N\setminus\{i\}})\right) \\
&= \frac{1}{n}\left(\mu(N,v)+\mu(N,w)\right) \\
&\quad +\frac{1}{n}\sum_{i\in N}\left(P^\mu(N\setminus\{i\}, v_{N\setminus\{i\}}) + P^\mu(N\setminus\{i\}, w_{N\setminus\{i\}})\right) \\
&= P^\mu(N,v) + P^\mu(N,w).
\end{aligned}$$

With equation (4) it then follows that $DP^\mu(N,v+w) = DP^\mu(N,v) + DP^\mu(N,w)$, and thus

$$\mu(N,v+w)NDP^\mu(N,v+w) = \mu(N,v)NDP^\mu(N,v) + \mu(N,w)NDP^\mu(N,w)$$

when both $(N,v),(N,w) \in \mathcal{C}^+$. If both games belong to $\mathcal{C}^0$ then additivity of $\mu$ implies that $\mu(N,v+w) = 0$, and thus again $\mu$-additivity holds. Finally, in case one of the games belong to $\mathcal{C}^0$ and the other to $\mathcal{C}^+$, say $(N,v) \in \mathcal{C}^0$ and $(N,w) \in \mathcal{C}^+$, then additivity of $\mu$ implies that $\mu(N,v+w) = \mu(N,w) > 0$, and as shown above $DP^\mu(N,v+w) = DP^\mu(N,v) + DP^\mu(N,w) = DP^\mu(N,w)$, and thus $\mu(N,v+w)NDP^\mu(N,v+w) = DP^\mu(N,v+w) = DP^\mu(N,w) = \mu(N,w)NDP^\mu(N,w) = \mu(N,v)NDP^\mu(N,v) + \mu(N,w)NDP^\mu(N,w)$. Hence, $\mu$-additivity holds for any pair $(N,v),\ (N,w) \in \mathcal{C}$. $\square$

Examples of additive, symmetric and null player independent $\mu$-functions are the functions $\mu^S(N,v) = v(N)$, $\mu^B(N,v) = \frac{1}{2^{n-1}}\sum_{E\subset N}(2|E|-n)v(E)$ and $\mu^M(N,v) =$

$\sum_{i \in N} m_N^i(N,v)$ as defined in the previous section. So, Theorem 3.3 holds for $\rho^S$, $\rho^B$ and $\rho^M$. The function $\mu^T(N,v) = \sum_{E \subset N} v(E)$ is additive and symmetric, but not null player independent. However, Theorem 3.3 also holds for $\rho^T$ since this share function also can be obtained using the null player independent function $\overline{\mu}^T(N,v) = \frac{1}{2^{n-1}}\mu^T(N,v)$.

**Corollary 3.4** *The share functions $\rho^S$, $\rho^B$, $\rho^T$ and $\rho^M$ are equal to the normalized marginal functions $NDP^{\mu^S}$, $NDP^{\mu^B}$, $NDP^{\overline{\mu}^T}$ and $NDP^{\mu^M}$ respectively.*

Observe that the Shapley share function $\rho^S$ is equal to the normalized marginal function $NDP^{\mu}$ on $\mathcal{G}_\mu$ by taking $\mu(N,v) = v(N)$ in Definition 3.1 of the $\mu$-potential function. From this it follows that for every share function corresponding to an additive, symmetric, and null player independent $\mu$-function it holds that the vector of shares of a game $(N,v) \in \mathcal{G}_\mu$ is equal to the vector of Shapley shares of the transformed game $(N,v^\mu)$ defined by $v^\mu(E) = \mu(E,v_E)$ for all $E \subset N$.

**Theorem 3.5** *Let $\mu: \mathcal{G} \to \mathbb{R}$ with $\mu(N,v^0) = 0$ for every set of players $N$, be additive, symmetric, and null player independent on $\mathcal{G}_\mu$, let $\mathcal{C} \subset \mathcal{G}_\mu$ be a subgame closed set contain all positively scaled unanimity games and let $\rho^\mu$ be the unique share function on $\mathcal{C}$ satisfying the properties of Corollary 2.2. Then for every $(N,v) \in \mathcal{C}$ it holds that $(N,v^\mu) \in \mathcal{G}_{\mu^S}$ and that $\rho^\mu(N,v) = \rho^S(N,v^\mu)$.*

PROOF

First, let $(N,v) \in \mathcal{C} \subset \mathcal{G}_\mu$. Since $\mu^S(N,v^\mu) = v^\mu(N) = \mu(N,v) \geq 0$, it follows that $(N,v^\mu) \in \mathcal{G}_{\mu^S}$. Next we show that $\rho^\mu(N,v) = \rho^S(N,v^\mu)$. First consider the case that $(N,v)$ is a null game and thus also all subgames $(E,v_E)$, $E \subset N$, are null games. Then by definition $\rho_i^\mu(N,v) = \frac{1}{n}$ for all $i \in N$. Furthermore we have that $(N,v^\mu)$ is a null game because $\mu(E,v_E) = 0$ for all $E \subset N$ and hence also $\rho_i^S(N,v^\mu) = \frac{1}{n}$ for all $i \in N$.

To prove the theorem for all games in $\mathcal{C}$ we show that the function $\rho$ defined by $\rho(N,v) = \rho^S(N,v^\mu)$ for all $E \subset N$ is a share function and satisfies symmetry and $\mu$-additivity on $\mathcal{C}$, and the null player property on $\mathcal{C}^+$. Since the share function satisfying these properties is unique we must then have that $\rho^\mu(N,v) = \rho(N,v)$ for any $(N,v) \in \mathcal{C} \subset \mathcal{G}_\mu$. That $\rho$ is a share function follows immediately from the fact that $\rho^S$ is a share function. Let $i \in N$ be a null player in $(N,v) \in \mathcal{C}^+$. Then the assumption of null player independence of $\mu$ implies that $i$ is a null player in $(N, v^\mu)$, and thus $\rho_i(N,v) = \rho_i^S(N,v^\mu) = 0$. Further, let $i,j \in N$ be symmetric players in $(N,v) \in \mathcal{C}$. Then symmetry of $\mu$ implies that $i$ and $j$ are symmetric in $(N,v^\mu)$, and thus $\rho_i(N,v) = \rho_i^S(N,v^\mu) = \rho_j^S(N,v^\mu) = \rho_j(N,v)$.

To show the $\mu$-additivity of $\rho$, note that for $(N,v),(N,w) \in \mathcal{C}$, by the additivity of $\mu$ and using $(v+w)^\mu = v^\mu + w^\mu$ we obtain that

$$\mu(N,v+w)\rho(N,v+w) = \mu(N,v+w)\rho^S(N,(v+w)^\mu) = \mu(N,v+w)\rho^S(N,v^\mu+w^\mu).$$

If at least one of the games $(N,v),(N,w)$ belong to $\mathcal{C}^+$ we obtain by $\mu^S$-additivity of $\rho^S$, and the fact that $\mu^S(N,(v+z)^\mu) = (v+z)^\mu(N) = \mu(N,v+z) > 0$, that

$$\begin{aligned}
\mu(N,v+w)\rho(N,v+w) &= \mu(N,v+w)\left(\frac{\mu^S(N,v^\mu)\rho^S(N,v^\mu) + \mu^S(N,w^\mu)\rho^S(N,w^\mu)}{\mu^S(N,(v+z)^\mu)}\right) \\
&= \mu(N,v+w)\left(\frac{v^\mu(N)\rho(N,v) + w^\mu(N)\rho(N,w)}{(v+z)^\mu(N)}\right) \\
&= \mu(N,v+w)\left(\frac{\mu(N,v)\rho(N,v) + \mu(N,w)\rho(N,w)}{\mu(N,v+w)}\right) \\
&= \mu(N,v)\rho(N,v) + \mu(N,w)\rho(N,w).
\end{aligned}$$

In case that both $(N,v),(N,w) \in \mathcal{C}^0$, additivity of $\mu$ and $\mu(N,v) = \mu(N,w) = 0$ imply that $\mu(N,v+w) = 0$, and thus $\mu$-additivity is satisfied. $\square$

We illustrate this theorem with an example.

**Example 3.6** Let $(N,v) \in \mathcal{G}$ be given by $N = \{1,2,3\}$ and $v = u^{\{1,2\}} + u^{\{1,2,3\}}$, i.e.

$$v(E) = \begin{cases} 0 & \text{if } E \in \{\{1\},\{2\},\{3\},\{1,3\},\{2,3\}\} \\ 1 & \text{if } E \in \{\{1,2\}\} \\ 2 & \text{if } E = \{1,2,3\}. \end{cases}$$

The various $\mu$-functions are represented in Table 1. Applying Theorem 3.5 yields

$$\rho^S(N,v) = \frac{1}{12}(5,5,2)$$

| $\mu(E,v_E)$ | $\{1\}$ | $\{2\}$ | $\{3\}$ | $\{1,2\}$ | $\{1,3\}$ | $\{2,3\}$ | $\{1,2,3\}$ |
|---|---|---|---|---|---|---|---|
| $\mu^S(E,v_E)$ | 0 | 0 | 0 | 1 | 0 | 0 | 2 |
| $\mu^B(E,v_E)$ | 0 | 0 | 0 | 1 | 0 | 0 | $\frac{7}{4}$ |
| $\mu^T(E,v_E)$ | 0 | 0 | 0 | 1 | 0 | 0 | 3 |
| $\overline{\mu}^T(E,v_E)$ | 0 | 0 | 0 | $\frac{1}{2}$ | 0 | 0 | $\frac{3}{4}$ |
| $\mu^M(E,v_E)$ | 0 | 0 | 0 | 2 | 0 | 0 | 5 |

Table 1: $\mu$-functions of Example 3.7

$$\begin{aligned}\rho^S(N,v^{\mu^B}) &= \frac{1}{7}(3,3,1) = \rho^B(N,v)\\ \rho^S(N,v^{\overline{\mu}^T}) &= \frac{1}{9}(4,4,1) = \rho^T(N,v)\\ \rho^S(N,v^{\mu^M}) &= \frac{1}{5}(2,2,1) = \rho^M(N,v).\end{aligned}$$

Note that $\rho^S(N,v^{\mu^T}) = \frac{1}{18}(7,7,4) \neq \rho^T(N,v)$.

In Hart and Mas-Colell (1988, 1989) it is shown that the potential function $P\colon \mathcal{G} \to \mathbb{R}$ as defined in the beginning of this section is given by

$$P(N,v) = \sum_{E \subset N} \frac{(n-|E|)!(|E|-1)!}{n!} v(E),$$

i.e. $P(N,v)$ is a weighted sum of the payoffs given in the characteristic function and where the weights are given by the Shapley weights in the Shapley $\mu$-function $\mu^S(N,v) = v(N) = \sum_{i \in N} \sum_{\substack{E \subset N \\ E \ni i}} \frac{(n-|E|)!(|E|-1)!}{n!} m_E^i(N,v)$ written as the weighted sum of all marginal contributions. Theorem 3.5 enables us to generalize the result of Hart and Mas-Colell for $\mu$-functions being a weighted sum of the marginal contributions with positive weights $\omega^n_{|E|}$, $E \subset N$, i.e. for functions $\sigma^{\omega^n}\colon \mathcal{G} \to \mathbb{R}$ as defined in Theorem 2.1 and given by $\sigma^{\omega^n}(N,v) = \sum_{i \in N} \sum_{\substack{E \subset N \\ E \ni i}} \omega^n_{|E|} m_E^i(N,v)$ for a vector $\omega^n$ of positive weights. Note that such a $\sigma^{\omega^n}$-function is additive and symmetric on $\mathcal{G}$ by definition.

The next corollary shows that the $\sigma^{\omega^n}$-potential function corresponding to a null player independent function $\sigma^{\omega^n}$ can be seen as a weighted sum of worths of coalitions with corresponding marginal contribution weights $\omega^n_{|E|}$, $E \subset N$. The corollary follows directly from Theorem 3.5 and the result of Hart and Mas-Colell.

**Corollary 3.7** *For given vector $\omega^n = (\omega^n_1, \ldots, \omega^n_n)$, $n \in \mathbb{N}$, of positive weights, let $\sigma^{\omega^n}$ be given by $\sigma^{\omega^n}(N,v) = \sum_{i\in N} \sum_{\substack{E\subset N \\ E\ni i}} \omega^n_{|E|} m^i_E(N,v)$. If $\sigma^{\omega^n}$ is null player independent on a subgame closed set $\mathcal{C} \subset \mathcal{G}_{\sigma^{\omega^n}}$, then the $\sigma^{\omega^n}$-potential function $P^{\sigma^{\omega^n}}$ on $\mathcal{C}$ is given by*

$$P^{\sigma^{\omega^n}}(N,v) = \sum_{E\subset N} \omega^n_{|E|} v(E).$$

# 4 Reduced games

For a given value function $f$ on $\mathcal{G}$, Hart and Mas-Colell (1988, 1989) have introduced their concept of a reduced game of $(N,v)$ for any nonempty coalition $T \subset N$. To distinguish this notion from other concepts of reduced games we will speak about the HM-$f$-reduced game. For $(N,v) \in \mathcal{G}$ and $T \subset N$ nonempty, the HM-$f$-reduced game of coalition $T$ is defined implicitly as the game $(T, v^{T,f})$ satisfying the conditions $v^{T,f}(\emptyset) = 0$ and

$$v^{T,f}(E) = v(E \cup T^c) - \sum_{j\in T^c} f_j(E \cup T^c, v_{E\cup T^c}),\ E \subset T,\ E \neq \emptyset,$$

where $T^c = N \setminus T$. Using this concept of reduced game, Hart and Mas-Colell (1988, 1989) stated an alternative characterization of the Shapley value. A characterization of the Banzhaf value in terms of a similar kind of reduced game has been given in Dragan (1996a,b).

In this section we will give similar characterizations for the Shapley and Banzhaf share functions. First, for given share function $\rho$ on $\mathcal{G}$ and function $\mu: \mathcal{G} \to \mathbb{R}$, the concept of HM-$f$-reduced game is generalized by defining implicitly for each $(N,v) \in \mathcal{G}$ and nonempty set $T \subset N$ the *HM-$(\mu,\rho)$-reduced game* $(T, v^{T,\mu,\rho})$ of coalition $T$ as the game that satisfies $v^{T,\mu,\rho}(\emptyset) = 0$ and

$$\mu(E, v^{T,\mu,\rho}_E) = \mu(E \cup T^c, v_{E\cup T^c}) \left(1 - \sum_{j\in T^c} \rho_j(E \cup T^c, v_{E\cup T^c})\right),\ E \subset T,$$

when $E \neq \emptyset$. Since $\rho$ is a share function we have that $\sum_{i\in F} \rho_i(F, v_F) = 1$ for every game $(F, v_F)$. Applying this with $F = E \cup T^c$ this equation can be rewritten as

$$\mu(E, v_E^{T,\mu,\rho}) = \mu(E \cup T^c, v_{E\cup T^c}) \sum_{j\in E} \rho_j(E \cup T^c, v_{E\cup T^c}),\ \ E \subset T. \tag{7}$$

Before characterizing share functions using reduced game properties we first have to discuss certain properties of HM-$(\mu,\rho)$-reduced games. From Hart and Mas-Colell (1989) it follows that for $\mu = \mu^S$ and $\rho = \rho^S$ the HM-$(\mu^S,\rho^S)$-reduced games $(T, v^{T,\mu^S,\rho^S})$ exist and are uniquely determined on $\mathcal{G}$. However, both the existence and uniqueness result do not hold for arbitrary chosen share function $\rho$ on $\mathcal{G}$ and function $\mu\colon \mathcal{G} \to \mathbb{R}$, as is shown in the next examples.

**Example 4.1** Take $\rho = \rho^S$ the Shapley share function and let $\mu\colon \mathcal{G} \to \mathbb{R}$ be given by $\mu(N, v) = \sum_{i\in N} v(\{i\})$. Consider the game $(N, v) \in \mathcal{G}$ given by $N = \{1, 2, 3\}$, $v = u^{\{3\}} + u^{\{1,2\}}$, so that $\mu(N, v) = 1$, and take $T = \{1, 2\}$. The HM-$(\mu,\rho^S)$-reduced game $(T, v^{T,\mu,\rho^S})$ must satisfy

$$v^{T,\mu,\rho^S}(\{i\}) = \mu(\{i\}, v_{\{i\}}^{T,\mu,\rho^S}) = \mu(\{i,3\}, v_{\{i,3\}})\rho_i^S(\{i,3\}, v_{\{i,3\}}) = 1 \times 0 = 0,\ \ i = 1, 2,$$

and

$$\sum_{i\in T} v^{T,\mu,\rho^S}(\{i\}) = \mu(T, v^{T,\mu,\rho^S}) = \mu(N, v)\left(\rho_1^S(N, v) + \rho_2^S(N, v)\right) = 1 \times \frac{1}{2} = \frac{1}{2}.$$

Clearly, this contradicts the existence of a HM-$(\mu,\rho^S)$-reduced game $(T, v^{T,\mu,\rho^S})$.

**Example 4.2** Take $\rho = \rho^S$ the Shapley share function and let $\mu\colon \mathcal{G} \to \mathbb{R}$ be given by $\mu(N, v) = \max_{E\subset N} v(E)$. Consider the game $(N, v) \in \mathcal{G}$ given by $N = \{1, 2, 3\}$, $v = u^{\{3\}}$, so that $\mu(N, v) = 1$, and take $T = \{1, 2\}$. The HM-$(\mu,\rho^S)$-reduced game $(T, v^{T,\mu,\rho^S})$ must satisfy

$$\mu(\{i\}, v_{\{i\}}^{T,\mu,\rho^S}) = \mu(\{i,3\}, v_{\{i,3\}})\rho_i^S(\{i,3\}, v_{\{i,3\}}) = 1 \times 0 = 0 \text{ for } i = 1, 2,$$

and

$$\mu(T, v^{T,\mu,\rho^S}) = \mu(N, v)\left(\rho_1^S(N, v) + \rho_2^S(N, v)\right) = 1 \times 0 = 0.$$

Clearly, all games $(T, v)$ with $v(\{i\}) = 0$, $i = 1, 2$ and $v(T) = v(\{1, 2\}) \le 0$ satisfy these conditions and hence the HM-$(\mu, \rho^S)$-reduced game $(T, v^{T,\mu,\rho^S})$ is not uniquely determined.

Thus, in general HM-$(\mu, \rho)$-reduced games need not exist nor be unique. However, it turns out that they are uniquely determined on $\mathcal{G}$ if $\mu$ is linear on $\mathcal{G}$ and is positive for all unanimity games.

**Theorem 4.3** *Let $\rho$ be a share function on $\mathcal{G}$ and let $\mu\colon \mathcal{G} \to \mathbb{R}$ be linear on $\mathcal{G}$ and positive for all unanimity games. Then for all $(N, v) \in \mathcal{G}$ and $T \subset N$, the HM-$(\mu, \rho)$-reduced games $(T, v^{T,\mu,\rho})$ are uniquely determined on $\mathcal{G}$.*

Proof

For given $(N, v) \in \mathcal{G}$, $T \subset N$, a share function $\rho$ on $\mathcal{G}$ and a linear function $\mu$ on $\mathcal{G}$ being positive for all unanimity games, let $(T, v^{T,\mu,\rho}) \in \mathcal{G}$ be a HM-$(\mu, \rho)$-reduced game for the coalition $T$. To show the existence and uniqueness of $(T, v^{T,\mu,\rho})$, we prove by induction on $|E|$ that there exist unique dividends $\Delta_{v^{\mu,T,\rho}}(E)$, $E \subset T$.

When $|E| = 1$ then linearity of $\mu$ implies that $\mu(E, v_E^{T,\mu,\rho}) = \mu(E, \Delta_{v_E^{T,\mu,\rho}} \cdot u^E) = \Delta_{v_E^{T,\mu,\rho}}(E)\mu(E, u^E)$. Since $\mu(E, u^E) > 0$ by assumption, it holds that the dividend $\Delta_{v^{T,\mu,\rho}}(E) = \Delta_{v_E^{T,\mu,\rho}}(E) = \frac{\mu(E, v_E^{T,\mu,\rho})}{\mu(E, u^E)}$ is uniquely determined. Proceeding by induction, assume that for some given integer $k \ge 1$ and for any $F \subset T$ with $|F| \le k$ we have determined the dividends $\Delta_{v^{T,\mu,\rho}}(F)$, and let $E \subset T$ be such that $|E| = k + 1$. By definition any HM-$(\mu, \rho)$-reduced game $(T, v^{T,\mu,\rho})$ satisfies

$$\mu(E, v_E^{T,\mu,\rho}) = \mu(E \cup T^c, v_{(E \cup T^c)}) \left( \sum_{j \in E} \rho_j(E \cup T^c, v_{(E \cup T^c)}) \right).$$

Linearity of $\mu$ implies that $\mu(E, v_E^{T,\mu,\rho}) = \sum_{F \subset E} \Delta_{v_E^{T,\mu,\rho}}(F)\mu(E, u^F)$, and by $\mu(E, u^E) > 0$ it then holds that

$$\Delta_{v_E^{T,\mu,\rho}}(E) = \frac{\mu(E, v_E^{T,\mu,\rho}) - \sum_{\substack{F \subset E \\ F \neq E}} \Delta_{v_E^{T,\mu,\rho}}(F)\mu(E, u^F)}{\mu(E, u^E)} =$$

$$\frac{\mu(E \cup T^c, v_{(E \cup T^c)}) \left( \sum_{j \in E} \rho_j(E \cup T^c, v_{(E \cup T^c)}) \right) - \sum_{\substack{F \subset E \\ F \neq E}} \Delta_{v_E^{T,\mu,\rho}}(F)\mu(E, u^F)}{\mu(E, u^E)}.$$

Since $\Delta_{v_E^{T,\mu,\rho}}(F) = \Delta_{v_F^{T,\mu,\rho}}(F)$ for all $F \subset E$, the induction hypothesis implies that $\Delta_{v_E^{T,\mu,\rho}}(E)$ is uniquely determined. Hence all the dividends $\Delta_{v^{T,\mu,\rho}}(E) = \Delta_{v_E^{T,\mu,\rho}}(E)$ exist and are uniquely determined and therefore it holds that all the values $v^{T,\mu,\rho}(E) = \sum_{F \subset E} \Delta_{v^{\mu,T,\rho}}(E) u^F(E)$ exist and are uniquely determined for all $E \subset T$, and so are $(T, v^{T,\mu,\rho})$ for all $T \subset N$. □

Theorem 4.3 implies that all HM-$(\mu,\rho)$-reduced games $(T, v^{T,\mu,\rho})$ are uniquely determined in $\mathcal{G}$ for all $(N,v)$ in the class of games considered in the next lemma which expresses the $\mu$-potential funtions of the subgames of a HM-$(\mu,\rho)$-reduced game as the difference of potential functions of subgames of the original game. (This lemma will be used later on to show the 'consistency' of certain share functions.)

**Lemma 4.4** *Let $\mu: \mathcal{G} \to \mathbb{R}$ with $\mu(N, v^0) = 0$ for every set of players $N$, be linear, symmetric, null player independent and positive for all unanimity games, and let $\rho^\mu$ be the corresponding share function as given in Corollary 2.2 on a subgame closed class $\mathcal{C} \subset \mathcal{G}_\mu$ containing all positively scaled unanimity games. Then for all $(N,v) \in \mathcal{C}$, $T \subset N$ and $E \subset T$ such that $\mu(E \cup T^c, v_{(E\cup T^c)}) > 0$ it holds that*

$$P^\mu(E, v_E^{T,\mu,\rho^\mu}) = P^\mu(E \cup T^c, v_{E\cup T^c}) - P^\mu(T^c, v_{T^c}). \tag{8}$$

Proof

We prove the lemma by induction on $|E|$. If $|E| = 0$ then both sides of equality (8) are equal to 0. If $|E| = 1$, i.e. $E = \{i\}$, then equations (3) and (7) and $P^\mu(\emptyset, v) = 0$ imply that

$$P^\mu(E, v_E^{T,\mu,\rho^\mu}) = \mu(E, v_E^{\mu,T,\rho^\mu}) = \mu(E \cup T^c, v_{(E\cup T^c)}) \rho_i^\mu(E \cup T^c, v_{(E\cup T^c)}).$$

Applying Theorem 3.3 and using the equations (4) and (5) this yields

$$\begin{aligned} P^\mu(E, v_E^{T,\mu,\rho^\mu}) &= \mu(E \cup T^c, v_{(E\cup T^c)}) NDP_i^\mu(E \cup T^c, v_{(E\cup T^c)}) \\ &= DP_i^\mu(E \cup T^c, v_{(E\cup T^c)}) = P^\mu(E \cup T^c, v_{(E\cup T^c)}) - P^\mu(T^c, v_{T^c}). \end{aligned}$$

Proceeding by induction assume that for some given integer $k \geq 1$ and for any $E' \subset T$ with $|E'| = k$ we have shown that the equality holds for $(E', v^{\mu,T,\rho^\mu})$, and let $|E| = k+1$. For every $i \in E$ the induction hypothesis then implies that

$$P^\mu(E \setminus \{i\}, v_{E\setminus\{i\}}^{T,\mu,\rho^\mu}) = P^\mu((E \cup T^c) \setminus \{i\}, v_{(E\cup T^c)\setminus\{i\}}) - P^\mu(T^c, v_{T^c}).$$

Again, with equations (3), (7), (4), (5) and $P^\mu(\emptyset, v) = 0$ it then follows that

$$|E|\left(P^\mu(E, v_E^{T,\mu,\rho^\mu}) - P^\mu(E \cup T^c, v_{E\cup T^c})\right) = \sum_{i\in E}\left(P^\mu(E, v_E^{T,\mu,\rho^\mu}) - P^\mu(E \cup T^c, v_{E\cup T^c})\right) =$$

$$= \sum_{i\in E}\left(P^\mu(E\setminus\{i\}, v_{E\setminus\{i\}}^{T,\mu,\rho^\mu})\right) + \mu(E, v_E^{T,\mu,\rho^\mu}) - \sum_{i\in E}\left(P^\mu((E\cup T^c)\setminus\{i\}, v_{(E\cup T^c)\setminus\{i\}})\right) -$$

$$\mu(E\cup T^c\setminus\{i\}, v_{E\cup T^c}) + \sum_{i\in T^c}\left(P^\mu(E\cup T^c, v_{E\cup T^c}) - P^\mu((E\cup T^c)\setminus\{i\}, v_{(E\cup T^c)\setminus\{i\}})\right) =$$

$$= \sum_{i\in E}\left(P^\mu(E\setminus\{i\}, v_{E\setminus\{i\}}^{T,\mu,\rho^\mu}) - P^\mu((E\cup T^c)\setminus\{i\}, v_{(E\cup T^c)\setminus\{i\}})\right) +$$

$$\mu(E\cup T^c, v_{E\cup T^c})\sum_{i\in E}\rho_i^\mu(E\cup T^c, v_{E\cup T^c})) - \mu(E\cup T^c, v_{E\cup T^c}) + \sum_{i\in T^c} DP_i(E\cup T^c, v_{E\cup T^c}) =$$

$$= \sum_{i\in E}\left(P^\mu(E\setminus\{i\}, v_{E\setminus\{i\}}^{T,\mu,\rho^\mu}) - P^\mu((E\cup T^c)\setminus\{i\}, v_{(E\cup T^c)\setminus\{i\}})\right) -$$

$$\mu(E\cup T^c, v_{E\cup T^c})\sum_{i\in T^c}\rho_i^\mu(E\cup T^c, v_{E\cup T^c}) + \sum_{i\in T^c} DP_i(E\cup T^c, v_{E\cup T^c}) =$$

$$= \sum_{i\in E} -P^\mu(T^c, v_{T^c}) = -|E|P^\mu(T^c, v_{T^c}),$$

so that $P^\mu(E, v_E^{T,\mu,\rho^\mu}) = P^\mu(E\cup T^c, v_{E\cup T^c}) - P^\mu(T^c, v_{T^c})$. □

For given function $\mu$ on $\mathcal{G}$, in the previous sections we have restricted ourselves to subclasses of $\mathcal{G}_\mu$ to characterize the corresponding share function $\rho^\mu$. Theorem 4.3 says that for given $\rho$ the HM-$(\mu,\rho)$-reduced game is uniquely determined if $\mu$ on $\mathcal{G}$ is linear and positive for all unanimity games. However, when we take $(N, v) \in \mathcal{G}_\mu$, Theorem 4.3 does not imply that also the corresponding HM-$(\mu,\rho)$-reduced games are in $\mathcal{G}_\mu$, as is illustrated in the following example [8].

[8]The same can be said about other classes of games such as the class of all monotone games. In a similar way as in the example below it can be shown that, for example, the HM-$(\mu^S, \rho^S)$-reduced game $(T, v^{T,\mu^S,\rho^S})$ with $T = \{1, 2\}$ of the monotone game $(N, v)$ given by $N = \{1, 2, 3\}$ and $v = u^{\{1,3\}} + u^{\{2,3\}} - u^{\{1,2,3\}}$ is not monotone.

**Example 4.5** Take $\rho = \rho^S$ the Shapley share function and $\mu(N,v) = \mu^S(N,v) = v(N)$. Consider the class $\mathcal{G}_{\mu^S}$ and $(N,v) \in \mathcal{G}_{\mu^S}$ given by $N = \{1,2,3\}$, $v = 2u^{\{3\}} - u^{\{1,2\}}$, so that $\mu^S(N,v) = 1$, and take $T = \{1,2\}$. The HM-$(\mu^S, \rho^S)$-reduced game $(T, v^{T,\mu^S,\rho^S})$ must satisfy

$$v^{T,\mu^S,\rho^S}(\{i\}) = v_{\{i\}}^{T,\mu^S,\rho^S}(\{i\}) = \mu^S(\{i\}, v_{\{i\}}^{T,\mu^S,\rho^S}) =$$

$$\mu^S(\{i,3\}, v_{\{i,3\}})\rho_i^S(\{i,3\}, v_{\{i,3\}}) = v_{\{i,3\}}(\{i,3\})\rho_i^S(\{i,3\}, v_{\{i,3\}}) = 2 \times 0 = 0, \;\; i = 1,2,$$

and

$$v^{T,\mu^S,\rho^S}(T) = v_T^{T,\mu^S,\rho^S}(T) = \mu^S(T, v^{T,\mu^S,\rho^S}) =$$

$$\mu^S(N,v)(\rho_1^S(N,v) + \rho_2^S(N,v)) = v(N)\left(\rho_1^S(N,v) + \rho_2^S(N,v)\right) = 1 \times (-1) = -1.$$

Thus, $\mu^S(T, v^{T,\mu^S,\rho^S}) < 0$, and $(T, v^{T,\mu^S,\rho^S}) \notin \mathcal{G}_{\mu^S}$.

For given $\mu$ and $\rho$, in the following we consider $(\mu,\rho)$-*closed* subclasses of the class of games $\mathcal{G}_\mu$.

**Definition 4.6** *For given $\mu$ and $\rho$ on $\mathcal{G}$, a subset $\mathcal{C}$ of $\mathcal{G}_\mu$ is $(\mu,\rho)$-***closed** *if it is subgame closed and for every $(N,v) \in \mathcal{C}$ and every $T \subset N$ it holds that the HM-$(\mu,\rho)$-reduced game $(T, v^{T,\mu,\rho}) \in \mathcal{C}$.*

This definition implies that when $\mathcal{C}$ is a $(\mu,\rho)$-closed subset of $\mathcal{G}$ for every $T \subset N$ it holds that $\mu(T, v_T) \geq 0$ and $\mu(E, v_E^{T,\mu,\rho}) \geq 0$ for all $E \subset T$. Recall from Theorem 4.3 that all HM-$(\mu,\rho)$-reduced games exist and are uniquely determined, when $\mu$ is linear on $\mathcal{G}$ and positive for all unanimity games. We now prove the following property.

**Lemma 4.7** *For $\mu$ linear on $\mathcal{G}$ and positive for all unanimity games, and share function $\rho$ on $\mathcal{G}$, let $\mathcal{C} \subset \mathcal{G}_\mu$ be $(\mu,\rho)$-closed. Then for every $(N,v) \in \mathcal{C}$ with $\mu(N,v) > 0$ it holds that $\rho_i(N,v) \geq 0$ for all $i \in N$.*

PROOF
Since $\mathcal{C}$ is $(\mu, \rho)$-closed for every $(N, v) \in \mathcal{C}$ we have that $(T, v^{T,\mu,\rho}) \in \mathcal{C}$ and hence $\mu(T, v^{T,\mu,\rho}) \geq 0$ for every $T \subset N$. For given $(N, v) \in \mathcal{G}$ with $\mu(N, v) > 0$, take $i \in N$. Applying equation (7) for $E = T = \{i\}$ yields

$$\mu(T, v^{T,\mu,\rho}) = \mu(N, v)\rho_i(N, v).$$

Hence $\rho_i(N, v) \geq 0$. $\square$

The proof of the lemma also implies that a subset $\mathcal{C} \subset \mathcal{G}_\mu$ can only be $(\mu, \rho)$-closed when for every $(N, v) \in \mathcal{C}$ with $\mu(N, v)$ positive the share vector is nonnegative. Clearly, when $\rho_i(N, v) < 0$ for some $i \in N$, then $\mu(\{i\}, v^{\{i\},\mu,\rho}) = \mu(N, v)\rho_i(N, v) < 0$ and hence $(\{i\}, v^{\{i\},\mu,\rho}) \notin \mathcal{C}$.

We now come to characterizing share functions using reduced game properties. We s-tate the following reduced game property, which says that for given function $\mu: \mathcal{G} \to \mathbb{R}$ and given share function $\rho$, a (possibly different) share function $\overline{\rho}$ satisfies the *HM-$(\mu, \rho)$-reduced game property* on a subgame closed subset $\mathcal{C}$ of $\mathcal{G}$ if for any $(N, v) \in \mathcal{C}$, for all $T \subset N$ and for every pair of two players $i,\ j \in T$, it holds that the ratio of the shares $\overline{\rho}_i$ and $\overline{\rho}_j$ in the HM-$(\mu, \rho)$-reduced game of coalition $T$ is equal to the ratio of their shares in the original game if $\sum_{j \in T} \overline{\rho}_j(N, v) \neq 0$. More precisely the property says:

**HM-$(\mu, \rho)$-reduced game property.** Let $\mu: \mathcal{G} \to \mathbb{R}$ be linear and positive for all unanimity games and let $\rho$ be a share function on $\mathcal{G}$. Then the share function $\overline{\rho}$ satisfies the HM-$(\mu, \rho)$-reduced game property on a subgame closed subset $\mathcal{C}$ of $\mathcal{G}_\mu$ if for every $(N, v) \in \mathcal{C}$ and every nonempty $T \subset N$ and $i \in T$, it holds that

$$\overline{\rho}_i(T, v^{T,\mu,\rho}) = \frac{\overline{\rho}_i(N, v)}{\sum_{j \in T} \overline{\rho}_j(N, v)} \quad \text{if} \quad \sum_{j \in T} \overline{\rho}_j(N, v) \neq 0.$$

Observe that we allow that $\rho$ and $\overline{\rho}$ are different ($\overline{\rho}$ satisfies the property stated above with respect to the reduced game that is obtained by using $\rho$ in equation (7)). The reduced game property is considered to be a consistency property when a function satisfies its own reduced game property, i.e. for given function $\mu$ a share function $\rho$ is considered to be consistent if $\rho$ satisfies its own HM-$(\mu, \rho)$-reduced game property. Now, for given $\mu$ linear and positive for all unanimity games and given share function $\rho$, let $\mathcal{C}$ be a $(\mu, \rho)$-closed subset of $\mathcal{G}$. Then Lemma 4.7 says that for any $i \in T$ and $(N, v) \subset \mathcal{C}$ it holds that $\rho_i(N, v) \geq 0$, i.e. $\rho$ is nonnegative on $\mathcal{C}$. Moreover this implies that $\rho_i(N, v) = 0$ for all $i \in T$ when $\sum_{j \in T} \rho_j(N, v) = 0$. We therefore also require that each player $i \in T$ gets an equal share in the own HM-$(\mu, \rho)$-reduced $(T, v^{T,\mu,\rho})$ when $\sum_{j \in T} \rho_j(N, v) = 0$. This gives the following consistent reduced game property.

**Axiom 4.8 (Consistent HM-$(\mu, \rho)$-reduced game property)** *Let $\mu: \mathcal{G} \rightarrow \mathbb{R}$ be linear and positive for all unanimity games. Then the share function $\rho$ satisfies the consistent HM-$(\mu, \rho)$-reduced game property on a $(\mu, \rho)$-closed subset $\mathcal{C}$ of $\mathcal{G}_\mu$ if for every $(N, v) \in \mathcal{C}$ and every nonempty $T \subset N$ and $i \in T$, it holds that*

$$\rho_i(T, v^{T,\mu,\rho}) = \begin{cases} \frac{\rho_i(N,v)}{\sum_{j \in T} \rho_j(N,v)} & \text{if } \sum_{j \in T} \rho_j(N, v) > 0, \\ \frac{1}{|T|} & \text{if } \sum_{j \in T} \rho_j(N, v) = 0. \end{cases}$$

The following consistency theorem says that for given $\mu$ the corresponding share function $\rho^\mu$ is consistent.

**Theorem 4.9** *Let $\mu: \mathcal{G} \rightarrow \mathbb{R}$ with $\mu(N, v^0) = 0$ for every set of players $N$, be linear, symmetric, null player independent and positive for all unanimity games. Then the share function $\rho^\mu$ on $\mathcal{G}_\mu$ as given in Corollary 2.2 satisfies the consistent HM-$(\mu, \rho^\mu)$-reduced game property on any $(\mu, \rho^\mu)$-closed subset $\mathcal{C}$ of $\mathcal{G}_\mu$ containing all positively scaled unanimity games.*

PROOF

Since $\mu$ is linear and positive for all unanimity games we have that all HM-$(\mu, \rho^\mu)$-reduced games exist and are uniquely determined. Since $\mathcal{C}$ is $(\mu, \rho^\mu)$-closed we have that $(T, v^{T,\mu,\rho^\mu}) \in \mathcal{C}$ and hence $\mu(T, v^{T,\mu,\rho}) \geq 0$ for every $T \subset N$. Now consider some game $(N, v) \in \mathcal{C}$. In case $\mu(N, v) = 0$ we have that $\rho_i^\mu(N, v) = \frac{1}{n}$ for all $i \in N$. Furthermore $\mu(N, v) = 0$ implies with equation (7) that $\mu(T, v^{\mu,T,\rho^\mu}) = 0$ and hence

$\rho_i^\mu(T, v^{\mu,T,\rho^\mu}) = \frac{1}{|T|}$ for all $i \in T$ for all $T \subset N$. So, $\rho^\mu(N,v)$ satisfies the consistent HM-$(\mu, \rho^\mu)$-reduced game property.

It remains to consider the case that $\mu(N,v) > 0$. Take $T \subset N$, $T \neq \emptyset$. If $\sum_{j\in T} \rho_j^\mu(N,v) = 0$, then it follows from equation (7) that $\mu(T, v^{\mu,T,\rho^\mu}) = 0$ and hence $\rho_i^\mu(T, v^{\mu,T,\rho^\mu}) = \frac{1}{|T|}$ for all $i \in T$, so that the consistent HM-$(\mu, \rho^\mu)$-reduced game property is satisfied. Otherwise $\sum_{j\in T} \rho_j^\mu(N,v) > 0$ and it follows from equation (7) that also $\mu(T, v^{\mu,T,\rho^\mu}) > 0$. Since $\mu(T \cup T^c, v_{T\cup T^c}) = \mu(N,v) > 0$, from Lemma 4.4 it follows that $DP_i^\mu(T, v^{T,\mu,\rho^\mu}) = P^\mu(T, v^{T,\mu,\rho^\mu}) - P^\mu(T \setminus \{i\}, v^{T,\mu,\rho^\mu}_{T\setminus\{i\}}) = P^\mu(T \cup T^c, v_{T\cup T^c}) - P^\mu(T^c, v_{T^c}) - P^\mu((T\setminus\{i\}) \cup T^c, v_{(T\setminus\{i\})\cup T^c}) + P^\mu(T^c, v_{T^c}) = P^\mu(N,v) - P^\mu(N\setminus\{i\}, v_{N\setminus\{i\}}) = DP_i^\mu(N,v)$. Applying Theorem 3.3, it follows for $i \in T$ that

$$\frac{\rho_i(N,v)}{\sum_{j\in T}\rho_j(N,v)} = \frac{DP_i^\mu(N,v)}{\sum_{j\in T} DP_j^\mu(N,v)} = \frac{DP_i^\mu(T, v^{T,\mu,\rho^\mu})}{\sum_{j\in T} DP_j^\mu(T, v^{T,\mu,\rho^\mu})} =$$
$$\frac{DP_i^\mu(T, v^{\mu,T,\rho^\mu})}{\mu(T, v^{T,\mu,\rho^\mu})} = NDP_i^\mu(T, v^{\mu,T,\rho^\mu}) = \rho_i^\mu(T, v^{\mu,T,\rho^\mu}).$$

Thus, $\rho^\mu(N,v)$ satisfies the consistent HM-$(\mu,\rho^\mu)$-reduced game property. □

Although for given $\mu$ the share function $\rho^\mu$ satisfies the consistent HM-$(\mu,\rho^\mu)$-reduced game property, this property is not sufficient to characterize $\rho^\mu$. Therefore we need an additional property. Hart and Mas-Colell (1988, 1989) characterize the Shapley value by their reduced game property and the property of *standardness for two person games*. Dragan (1996a,b) characterizes the Banzhaf value using a modified version of Hart and Mas-Colell's reduced game, standardness for two person games and *one player efficiency*. Here we use a generalization of the standardness for two person games property to $\mu$-standardness. Recall that $v = \sum_{T\subset N} \Delta_v(T) u^T$.

**Axiom 4.10 ($\mu$-standardness for two person games)** *Let $\mu: \mathcal{G} \to \mathbb{R}$ be linear and positive for all unanimity games and let $\mathcal{C}$ be a subset of $\mathcal{G}_\mu$ containing all positively scaled unanimity games. A share function $\rho$ is $\mu$-standard for two person games on $\mathcal{C}$ if for any two player game $(N,v) \in \mathcal{C}$ and for $i,j \in N$, $j \neq i$, it holds that*

$$\rho_i(N,v) = \begin{cases} \frac{1}{2}\left(\frac{\mu(N,v)+\mu(\{i\},v_{\{i\}})-\mu(\{j\},v_{\{j\}})}{\mu(N,v)}\right) & \text{if } \mu(N,v) > 0 \\ \frac{1}{2} & \text{if } \mu(N,v) = 0. \end{cases}$$

Observe that for the Shapley $\mu$-function $\mu^S(N,v) = v(N)$ the expression for $\rho_i(N,v)$ in case $v(N) > 0$ reduces to

$$\rho_i(N,v) = \frac{1}{2}\left(\frac{v(N) + v(\{i\}) - v(\{j\})}{v(N)}\right),$$

which corresponds to the property of standardness for two person games as used in Hart and Mas-Colell (1988, 1989). The next theorem states that for given $\mu$ the share function $\rho^\mu$ of Corollary 2.2 is $\mu$-standard for two person games.

**Theorem 4.11** *Let $\mu\colon \mathcal{G} \to \mathbb{R}$ be linear, symmetric, and null player independent on $\mathcal{G}$ and positive for all unanimity games. Then the share function $\rho^\mu$ as given in Corollary 2.2 is $\mu$-standard for two person games on any $(\mu, \rho^\mu)$-closed subset $\mathcal{C}$ of $\mathcal{G}_\mu$ containing all positively scaled unanimity games.*

Proof

By definition, $\rho^\mu$ satisfies the $\mu$-standardness for two person games for any two player game $(N,v) \in \mathcal{C}$ with $\mu(N,v) = 0$. Now, suppose $(N,v)$ is a game with $|N| = 2$ and $\mu(N,v) > 0$. Observe that $v$ can be written as $v = \sum_{T \subset N} \Delta_v(T)u^T = \sum_{T \in \mathcal{T}^+} \Delta_v(T)u^T - \sum_{T \in \mathcal{T}^-}(-\Delta_v(T))u^T$, where $\mathcal{T}^+ = \{T \subset N | \Delta_v(T) > 0\}$ and $\mathcal{T}^- = \{T \subset N | \Delta_v(T) < 0\}$. So, $v$ can de written as a linear combination of positively scaled unanimity games in $\mathcal{C}$. Since $\mu$ is linear and $\rho^\mu$ is $\mu$-additive and is well defined for all positively scaled unanimity games, it follows for $i \in N$ that

$$\mu(N,v)\rho_i^\mu(N,v) =$$

$$\sum_{T \in \mathcal{T}^+} \mu(N, \Delta_v(T)u^T)\rho_i(N, \Delta_v(T)u^T) - \sum_{T \in \mathcal{T}^-} \mu(N, -\Delta_v(T)u^T)\rho_i(N, -\Delta_v(T)u^T) =$$

$$\sum_{T \in \mathcal{T}^+} \mu(N, \Delta_v(T)u^T)\rho_i(N, \Delta_v(T)u^T) + \sum_{T \in \mathcal{T}^-} \mu(N, \Delta_v(T)u^T)\rho_i(N, -\Delta_v(T)u^T) \qquad (9)$$

The null player property of $\rho^\mu$ implies that for any $\alpha > 0$ it holds that $\rho_j^\mu(N, \alpha u^T) = 0$ when $j \notin T$ and symmetry implies that $\rho_i^\mu(N, \alpha u^T) = \frac{1}{|T|}$ when $i \in T$. Applying this for every nonempty $T \subset N$ for the two player game $(N,v)$ with $N = \{i,j\}$, we obtain for $i,j \in N$, $j \neq i$, that $\rho_i^\mu(N, |\Delta_v(\{i\})|u^{\{i\}}) = 1$, $\rho_i^\mu(N, |\Delta_v(\{j\})|u^{\{j\}}) = 0$ and

$\rho_i^\mu(N, |\Delta_v(N)|u^N) = \frac{1}{2}$. Hence, together with the linearity of $\mu$, equation (9) reduces to

$$\mu(N,v)\rho_i^\mu(N,v) = \mu(N, \Delta_v(\{i\})u^{\{i\}}) + \frac{1}{2}\mu(N, \Delta_v(N)u^N), \quad i \in N.$$

Since $\mu$ is linear and thus

$$\mu(N,v) = \mu(N, \Delta_v(\{i\})u^{\{i\}}) + \mu(N, \Delta_v(\{j\})u^{\{j\}}) + \mu(N, \Delta_v(N)u^N),$$

it follows that

$$\mu(N,v)\rho_i^\mu(N,v) =$$

$$\frac{1}{2}(\mu(N,v) + \mu(N, \Delta_v(\{i\})u^{\{i\}}) - \mu(N, \Delta_v(\{j\})u^{\{j\}}), \quad i \in N, \ j \in N, \ j \neq i.$$

Since $\{j\}$ is a null player in $(N, u^{\{i\}})$, it follows from the null player independency of $\mu$ and $\Delta_v(\{i\}) = v(\{i\})$ that $\mu(N, \Delta_v(\{i\})u^{\{i\}}) = \mu(N \setminus \{j\}, \Delta_v(\{i\})u^{\{i\}}) = \mu(\{i\}, v_{\{i\}})$. Analogously we have that $\mu(N, \Delta_v(\{j\})u^{\{j\}}) = \mu(\{j\}, v_{\{j\}})$ and thus

$$\mu(N,v)\rho_i^\mu(N,v) = \frac{1}{2}(\mu(N,v) + \mu(\{i\}, v_{\{i\}}) - \mu(\{j\}, v_{\{j\}}), \quad i \in N, \ j \in N, \ j \neq i.$$

Hence, $\rho^\mu$ is $\mu$-standard for two person games on $\mathcal{C}$. $\square$

The Theorems 4.9 and 4.11 show that $\rho^\mu$ satisfies the consistent reduced game property and $\mu$-standardness for two person games on any $(\mu, \rho^\mu)$-closed subset of $g_\mu$ containing all positively scaled unanimity games. The next theorem shows that a share function $\rho$ is uniquely determined by the consistent reduced game property and $\mu$-standardness for two person games.

**Theorem 4.12** *Let $\mu: \mathcal{G} \rightarrow \mathbb{R}$ with $\mu(N, v^0) = 0$ for every set of players $N$, be linear, symmetric, and null player independent on $\mathcal{G}$ and positive for all unanimity games. Let $\rho$ be a share function on $\mathcal{G}$ and $\mathcal{C}$ a $(\mu, \rho)$-closed subset of $\mathcal{G}_\mu$ containing all positively scaled unanimity games. Then $\rho$ is uniquely determined when $\rho$ satisfies the consistent HM-$(\mu, \rho)$-reduced game property and the $\mu$-standardness for two person games property on $\mathcal{C}$.*

PROOF

Suppose that $\rho$ is a share function on $\mathcal{G}$ satisfying the consistent reduced game property and $\mu$-standardness for two person games on $\mathcal{C}$. By definition of share function we have that $\rho(N, v) = 1$ if $|N| = 1$. If $|N| = 2$, then $\rho(N, v)$ is uniquely determined by the $\mu$-standardness for two person games. Now, let $(N, v) \in \mathcal{C}$ be such that $|N| \geq 3$. Because of Lemma 4.7 we must have that $\rho_i(N, v) \geq 0$ for all $i \in N$. Since $\mu$ satisfies the conditions of Theorem 4.3 all reduced games exist and belong to the $(\mu, \rho)$-closed set $\mathcal{C}$. When $\mu(N, v) = 0$, then it follows from equation (7) that $\mu(T, v^{T,\mu,\rho}) = 0$ for all $T \subset N$. So, for any pair $i,\ j \in N,\ i \neq j$, we have with $T = \{i, j\}$ that $\rho_i(T, v^{T,\mu,\rho}) = \rho_j(T, v^{T,\mu,\rho}) = \frac{1}{2}$ because of the $\mu$-standardness for two person games. From the consistent HM-$(\mu, \rho)$ reduced game property and the nonnegativity of $\rho(N, v)$ it then follows that $\rho_i(N, v) = \rho_j(N, v)$ for any pair $i,\ j \in N$ and hence $\rho_i(N, v) = \frac{1}{|N|}$ for all $i \in N$ and thus $\rho(N, v)$ is uniquely determined.

It remains to consider the case $\mu(N, v) > 0$. We now proceed by induction on $|N|$. For $|N| \leq 2$, $\rho(N, v)$ is uniquely determined. Suppose that for $k \geq 3$, $\rho(N, v)$ is uniquely determined for any $N$, such that $|N| = k - 1$. Now, suppose $|N| = k$. Since $\rho$ is a share function and thus $\sum_{i \in N} \rho_i(N, v) = 1$, at least one component of $\rho(N, v)$ is positive. So, for some $i \in N$, let $\rho_i(N, v) > 0$. Since $\mathcal{C}$ is $(\mu, \rho)$-closed, we have by Lemma 4.7 that $\rho(N, v)$ is nonnegative and hence it follows that $\rho_i(N, v) + \rho_j(N, v) > 0$ for all $j \in N' \equiv N \backslash \{i\}$. Now, take $T = \{i, j\}$ for some $j \in N'$. Then, applying equation (7) for $T = \{i, j\}$ and $E = \{i\}$ yields that

$$\mu(\{i\}, v_{\{i\}}^{T,\mu,\rho}) = \mu(N \setminus \{j\}, v_{N \setminus \{j\}}) \rho_i(N \setminus \{j\}), v_{N \setminus \{j\}}). \tag{10}$$

By the induction hypothesis we have that $\rho_i(N \setminus \{j\}), v_{N \setminus \{j\}})$ is uniquely determined and thus also $\mu(\{i\}, v_{\{i\}}^{T,\mu,\rho})$ is uniquely determined. Analogously we have that

$$\mu(\{j\}, v_{\{j\}}^{T,\mu,\rho}) = \mu(N', v_{N'}) \rho_j(N', v_{N'}) \tag{11}$$

is uniquely determined. Furthermore, we have that

$$\mu(T, v^{T,\mu,\rho}) = \mu(N, v)(\rho_i(N, v) + \rho_j(N, v)) > 0.$$

So, applying the $\mu$-standardness for two games to the game $(\{T\}, v^{T,\mu,\rho})$ with $T = \{i,j\}$ yields

$$\rho_i(T, v^{T,\mu,\rho}) = \frac{1}{2} + \frac{\mu(\{i\}, v^{T,\mu,\rho}_{\{i\}}) - \mu(\{j\}, v^{T,\mu,\rho}_{\{j\}})}{2\mu(T, v^{T,\mu,\rho})} =$$

$$\frac{1}{2} + \frac{\mu(\{i\}, v^{T,\mu,\rho}_{\{i\}}) - \mu(\{j\}, v^{T,\mu,\rho}_{\{j\}})}{2\mu(N,v)(\rho_i(N,v) + \rho_j(N,v))} \; . \tag{12}$$

By applying the consistent HM-$(\mu,\rho)$-reduced game property for $T = \{i,j\}$ we also have that

$$\rho_i(T, v^{T,\mu,\rho}) = \frac{\rho_i(N,v)}{\rho_i(N,v) + \rho_j(N,v)}. \tag{13}$$

So, from equations (12) and (13) it follows that for all $j \in N' = N \setminus \{i\}$ it holds that

$$\frac{1}{2} + \frac{\mu(\{i\}, v^{\{i,j\},\mu,\rho}_{\{i\}}) - \mu(\{j\}, v^{\{i,j\},\mu,\rho}_{\{j\}})}{2\mu(N,v)(\rho_i(N,v) + \rho_j(N,v))} = \frac{\rho_i(N,v)}{\rho_i(N,v) + \rho_j(N,v)},$$

which can be rewritten to the system of equations

$$\rho_j(N,v) = \rho_i(N,v) + \frac{\mu(\{j\}, v^{\{i,j\},\mu,\rho}_{\{j\}}) - \mu(\{i\}, v^{\{i,j\},\mu,\rho}_{\{i\}})}{\mu(N,v)}, \qquad j \in N \setminus \{i\}, \tag{14}$$

where for all $j \in N \setminus \{i\}$ the $\mu(\{j\}, v^{\{i,j\},\mu,\rho}_{\{j\}})$ and $\mu(\{i\}, v^{\{i,j\},\mu,\rho}_{\{i\}})$ are uniquely determined by (10) and (11). So, with the condition $\sum_{h \in N} \rho_h(N,v) = 1$ the system (14) has a unique solution. $\square$

Theorem 4.12 says that the the consistent HM-$(\mu,\rho)$-reduced game property and the $\mu$-standardness for two person games property uniquely determine the share function $\rho$ on a $(\mu,\rho)$-closed subset $\mathcal{C}$ of $\mathcal{G}_\mu$ containing all positively scaled unanimity games. On the other hand, the Theorems 4.9 and 4.11 show that $\rho^\mu$ satisfies both properties on any $(\mu,\rho^\mu)$-closed subset of $\mathcal{G}_\mu$ containing all positively scaled unanimity games. Hence we have the following corollary.

**Corollary 4.13** *Let $\mu: \mathcal{G} \to \mathbb{R}$ with $\mu(N, v^0) = 0$ for every set of players $N$, be linear, symmetric, and null player independent on $\mathcal{G}$ and positive for all unanimity games. Then the share function $\rho^\mu$ as given in Corollary 2.2 is the unique share function that satisfies the consistent HM-$(\mu, \rho^\mu)$-reduced game property and the $\mu$-standardness for two person games property on any $(\mu, \rho^\mu)$-closed subset $\mathcal{C}$ of $\mathcal{G}_\mu$ containing all positively scaled unanimity games.*

# 5 Concluding remarks

We conclude this paper by making some remarks on reduced game properties for share mappings. A *value mapping* on $\mathcal{C} \subset \mathcal{G}$ is a mapping $F$ that assigns to every $(N, v) \in \mathcal{C}$ a set of $n$-dimensional real vectors $F(N, v) \subset \mathbb{R}^n$ each representing a distribution of the payoffs over the players. A value mapping $F$ is efficient on $\mathcal{C} \subset \mathcal{G}$ if $F(N, v) \subset \{x \in \mathbb{R}^n \mid \sum_{i \in N} x_i = v(N)\}$ for every $(N, v) \in \mathcal{C}$. An example of an efficient value mapping is the *Core* denoted by $C(N, v)$ and given by

$$C(N, v) = \left\{ x \in \mathbb{R}^n \,\middle|\, \sum_{i \in N} x_i(N, v) = v(N), \text{ and } \sum_{i \in E} x_i(N, v) \geq v(E) \text{ for all } E \subset N \right\}.$$

A *share mapping* on $\mathcal{C} \subset \mathcal{G}$ is a mapping $R$ which assigns to every $(N, v) \in \mathcal{C}$ a set of share vectors $R(N, v) \subset \mathcal{S}^N := \{\rho \in \mathbb{R}^n \mid \sum_{i \in N} \rho_i = 1\}$ for every $(N, v) \in \mathcal{C}$.

The share mapping corresponding to the Core is the *Core share mapping* given by

$$R^C(N, v) = \left\{ \rho \in \mathcal{S}^N \,\middle|\, v(N) \sum_{i \in E} \rho_i \geq v(E) \text{ for all } E \subset N \right\}.$$

Note that $R^C c(N, v) = \mathcal{S}^N$ if $(N, v)$ is a *null game*, i.e. if $v = v^0$ with $v^0(E) = 0$ for all $E \subset N$. The core of a null game only contains the null vector with all components equal to zero. If we distribute $v^0(N) = 0$ then the shares assigned to the players do not matter. Thus, $R(N, v) = \mathcal{S}^N$ seems reasonable in this case.

In this paper a class of share functions that generalizes the Shapley share function and Banzhaf share function is characterized using a generalization of the reduced games as introduced Hart and Mas-Colell (1988, 1989) in characterizing the Shapley value (adapted to share functions), and used by Dragan (1996a,b) in characterizing the Banzhaf value. In a similar way the Core share mapping can be generalized as is done in van den Brink and van der Laan (1999). They characterize a class of share mappings containing the Core share mapping using a generalization of Davis and Maschler's reduced game property[9] as used by Peleg (1986) in characterizing the Core. With each share function in the class considered in the underlying paper there corresponds a share mapping in the class considered in van den Brink and van der Laan (1999). The Core share mapping coincides with what is called the *Shapley share core*. The class considered also contains the *Banzhaf share core* which is related to the Banzhaf share function is a similar way as the Shapley share core is related to the Shapley value.

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[9] A similar remark as in footnote 4 is applicable here.

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